The Z_Transform:

* Introduction:

Sampled-data systems in which one or more variables can change only at discrete instants of time. These instants, which are denoted by KT, where K is the running parameter ($K=0,1,2,\ldots\infty$), may specify the time at which some physical measurement is performed. The time interval between two discrete instants is taken to be sufficiently short so that the data for the time between these discrete instants can be approximated by simple interpolation.

Sampled-data systems arise in practice whenever the measurements necessary for control are obtained by computer which is time-shared by several plants that a control signal is sent out to each plant only periodically or whenever digital computer is used to perform computations necessary for control.

* Z-Transformation:

The role played by the z-transformation in discrete systems is similar to that of the Laplace transformation in continuous time systems.

where; $\mathcal{L}\,f(t) = F(s) = \int_0^\infty f(t)e^{-st}dt$, Define $\left(Z = e^{Ts}\right)$ then;

$$Z(f(t)) = \sum_{K=0}^{K=\infty} f(KT)Z^{-K}$$

EX: Find the z-transform of the unit-step function $1(t)$.

Since $z(f(t)) = \sum_{K=0}^{\infty} f(KT)z^{-K}$ then $z\{1(t)\} = \sum_{K=0}^{K=\infty} (1)z^{-K} = 1+z^{-1}+z^{-2}+z^{-3}+\cdots$ (power series)

thus $z\{1(t)\} = \frac{1}{1-z^{-1}}$ (backward Representation) or $= \frac{z}{z-1}$ (forward Representation)

Similarly: for $x(t) = e^{-at}$ $t>0$, then $z\{e^{-at}\} = \frac{z}{z-e^{-aT}}$,

and for $x(K) = a^K$, then $z\{x(K)\} = \frac{z}{z-a}$.

Question: prove that for $x(t)=t$ $x(z) = \frac{Tz}{(z-1)^2}$.

prove that $Z(\sin \omega t) = \dfrac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$, $Z(\cos \omega t) = \dfrac{z^2 - z \cos \omega T}{z^2 - 2z \cos \omega T + 1}$.

* Some Important properties of Z-transform:

- Multiplying by a Constant: if $X(z)$ is the z-transform of $x(t)$, then

$$Z[a x(t)] = a Z(x(t)) = a X(z)$$

- Linearity of the z-transform: for $x(k) = a f(k) + b g(k)$ then $X(z) = a F(z) + b G(z)$

- Multiplying by a^k: $Z[a^k x(k)] = X(a^{-1} z)$

- Shifting theorem: $Z[x(t - nT)] = z^{-n} X(z)$

Ex: the z-transform of unit step delayed by 1 sample is $\left(\dfrac{z}{z-1}\right) z^{-1} = \dfrac{1}{z-1}$

and that delayed by 4 sampling period is: $\left(\dfrac{z}{z-1}\right) z^{-4} = \dfrac{1}{z^3(z-1)}$.

- Initial Value theorem: if $x(t)$ has the z-transform $X(z)$ and if $\lim_{z \to \infty} X(z)$ exists then the initial value $x(0) = \lim_{z \to \infty} X(z)$

Ex: for $X(z) = \dfrac{(1 - e^{-T}) z^{-1}}{(1 - z^{-1})(1 - e^{-T} z^{-1})}$, $x(0) = 0$, Knowing that $x(t)$ for the given $X(z)$ is $x(t) = 1 - e^{-t}$ and $x(0) = 0$.

thus $\lim_{k \to 0} x(k) = \lim_{z \to \infty} X(z) =$ initial value

- Final value theorem: $\lim_{k \to \infty} x(k) = \lim_{z \to 1} [(1 - z^{-1}) X(z)] = \lim_{z \to 1} \left(\dfrac{z-1}{z} X(z)\right) = \lim_{z \to 1} [(z-1) X(z)]$

Ex: Determine the final value $x(\infty)$ of: $X(z) = \dfrac{1}{1 - z^{-1}} - \dfrac{1}{1 - e^{-aT} z^{-1}}$ $a > 0$

$$x(\infty) = \lim_{z \to 1} \left[(1 - z^{-1}) X(z) \right]$$

$$x(\infty) = \lim_{z \to 1} \left\{ (1 - z^{-1}) \left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT} z^{-1}} \right] \right\} = \lim_{z \to 1} \left(1 - \frac{1 - z^{-1}}{1 - e^{-aT} z^{-1}} \right) = 1$$

Recall: $x(t) = 1 - e^{-t}$ for $t \to \infty$ $\therefore x(\infty) = 1$.

* Obtaining Z-transform using partial Fraction Expansion Method:

whenever a function in (S) is given, the corresponding Z-transform may be obtained by first expanding the given function in (S) into partial fraction then combining the Z-transform of each partial fraction term.

Ex:- obtain the Z-transform of $X(S) = \frac{1}{s(s+1)}$

Solution: $X(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{(s+1)}$, solve for $A=1$, $B=-1$ then $X(s) = \frac{1}{s} - \frac{1}{s+1}$

hence; $X(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$

* Solving Difference Equations by Z-transform method:

The Solution of Difference Equations by Z-transform is of the same importance of using L-transform in solving Differential Equations.

Knowing that: $Z[x(k+1)] = zX(z) - zx(0)$

$$Z[x(k+2)] = z^2X(z) - z^2x(0) - zx(1)$$

In general: $Z\{x(k+m)\} = z^m X(z) - z^m x(0) - z^{m-1}x(1) - z^{m-2}x(2) - \cdots - zx(m-1)$

Ex: Solve the following difference equation using Z-transform method:

$$x(k+2) + 3x(k+1) + 2x(k) = 0 \quad , \text{ with } x(0)=0, x(1)=1$$

Taking the Z-transform of both sides, we obtain:

$$z^2X(z) - z^2x(0) - zx(1) + 3zX(z) - 3zx(0) + 2X(z) = 0$$

Substituting the given initial values and simplifying gives $X(z) = \frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)}$

then $X(z) = \frac{z}{z+1} - \frac{z}{z+2}$ (by P.F.M)

Since $Z\{a^k\} = z/(z-a)$ then $x(k) = (-1)^k - (-2)^k$ for $(k = 0, 1, 2, 3, \ldots \infty)$.

* The Inverse Z-Transformation:

Given $X(z)$, there are 3-method of obtaining the inverse Z-transform to find $x(k)$ or $x(kT)$. The methods are (1. Expansion using infinite power series by long division, 2. partial Fraction Expansion method, 3- the inversion Integral).

Ex: Find $X(KT)$ for $K=0,1,2,3,4$ when $X(z)$ is given by $X(z)=\dfrac{10z}{(z-1)(z-2)}$

Solution: $X(z)=\dfrac{10z}{z^2-3z+2}=\dfrac{10z^{-1}}{1-3z^{-1}+2z^{-2}}$ using long division then we obtain:

$X(z)=10z^{-1}+30z^{-2}+70z^{-3}+150z^{-4}+\cdots$ then $X(0)=0$, $X(T)=10$, $X(2T)=30$,

$X(3T)=70$, $X(4T)=150$ --- etc.

using partial Fraction: $\dfrac{X(z)}{z}=\dfrac{10}{(z-1)(z-2)}=\dfrac{A}{(z-1)}+\dfrac{B}{(z-2)}$ then $A=-10$, $B=10$

hence $X(z)=\dfrac{-10z}{(z-1)}+\dfrac{10z}{(z-2)}$ then $X(KT)=10(-1+2^K)$ with $K=0,1,2,3,\cdots\infty$

thus $X(0)=0$, $X(T)=10$, $X(2T)=30$, $X(3T)=70$, $X(4T)=150$. find $X(30T)=?$

Questions: for above example answer the following:

1. why should we use Backward representation in applying long division?
2. which is better long division or P.F.E. method?
3. Is there any case that one of the methods above can not be applied easily?
4. what will happen if $X(z)=\dfrac{10}{(z-1)(z-2)}$?

(Problems)

1. Prove that the z-transform of A^K "where A is an $n\times n$ matrix" is $(zI-A)^{-1}z$
2. It is required to solve the following difference equation using z-transform:

$$m(K)=e(K)-e(K-1)-m(K-1) \quad \text{with} \quad \begin{matrix} e(K)=1 & \text{for } K \text{ is even} \\ e(K)=0 & \text{for } K \text{ is odd}\end{matrix}$$

Ans: $m(0)=1$, $m(1)=-2$, $m(2)=3$, $m(3)=-4$ --- etc.

3. for $E(z)=\dfrac{z}{z^2-6z+5}$ find $e(K)$ for $K=0,1,2,3,4$, using z-method. what Conclusion can you make? Ans: 0, 1, 6, 31, 156 ---- etc. find $e(0)$, $e(\infty)$ check your results.
4. Solve using z-transform $X(K)-3X(K-1)+2X(K-2)=e(K)$ with $X(-2)=X(-1)=0$

where $\begin{matrix} e(K)=1 & \text{for } K=0,1 \\ e(K)=0 & \text{for } K\geq 2\end{matrix}$ Ans: $X(K)=3(2)^K-2$ find $X(0)$, $X(\infty)$ check your answer.

5. Given $E(z)=\dfrac{1}{z^2-1.3z+0.3}$ find $E(K)$ using P.F.E. method.

6. A digital filter is described by $\dfrac{M(z)}{E(z)}=\dfrac{2z^2-3.5z+1.9}{z^3-2.7z^2+z-0.95}$ find the difference equation that describes the filter then draw its simulation diagram.

Entry	Laplace Domain	Time Domain	Z Domain (t=kT)
unit impulse	1	$\delta(t)$ unit impulse	1
unit step	$\Gamma(s)=\frac{1}{s}$	$\gamma(t)$ (note)	$\frac{z}{z-1}$
ramp	$\frac{1}{s^2}$	t	$T\frac{z}{(z-1)^2}$
parabola	$\frac{2}{s^3}$	t^2	$T^2\frac{z(z+1)}{(z-1)^3}$
t^n (n is integer)	$\frac{n!}{s^{(n+1)}}$	t^n	
exponential	$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$
power		b^k $\quad (b=e^{-aT})$	$\frac{z}{z-b}$
time multiplied exponential	$\frac{1}{(s+a)^2}$	te^{-at}	$T\frac{ze^{-aT}}{(z-e^{-aT})^2}$
Asymptotic exponential	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$	$\frac{z(1-e^{-aT})}{a(z-1)(z-e^{-aT})}$
double exponential	$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-at}-e^{-bt}}{(b-a)}$	$\frac{z(e^{-aT}-e^{-bT})}{(b-a)(z-e^{-aT})(z-e^{-bT})}$
asymptotic double exponential	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left(1-\frac{be^{-at}-ae^{-bt}}{(b-a)}\right)$	
asymptotic critically damped	$\frac{1}{s(s+a)^2}$	$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	
differentiated critically damped	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
sine	$\frac{\omega_0}{s^2+\omega_0^2}$	$\sin(\omega_0 t)$	$\frac{z\sin(\omega_0 T)}{z^2-2z\cos(\omega_0 T)+1}$
cosine	$\frac{s}{s^2+\omega_0^2}$	$\cos(\omega_0 t)$	$\frac{z(z-\cos(\omega_0 T))}{z^2-2z\cos(\omega_0 T)+1}$
decaying sine	$\frac{\omega_d}{(s+a)^2+\omega_d^2}$	$e^{-at}\sin(\omega_d t)$	$\frac{ze^{-aT}\sin(\omega_d T)}{z^2-2ze^{-aT}\cos(\omega_d T)+e^{-2aT}}$
decaying cosine	$\frac{s+a}{(s+a)^2+\omega_d^2}$	$e^{-at}\cos(\omega_d t)$	$\frac{z(z-e^{-aT}\cos(\omega_d T))}{z^2-2ze^{-aT}\cos(\omega_d T)+e^{-2aT}}$

Properties of Laplace Transform

Property Name	Illustration
Definition	$f(t) \overset{\mathcal{L}}{\leftrightarrow} F(s)$ $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$
Linearity	$Af_1(t) + Bf_2(t) \overset{\mathcal{L}}{\leftrightarrow} AF_1(s) + BF_2(s)$
First Derivative	$\frac{df(t)}{dt} \overset{\mathcal{L}}{\leftrightarrow} sF(s) - f(0^-)$
Second Derivative	$\frac{d^2f(t)}{dt^2} \overset{\mathcal{L}}{\leftrightarrow} s^2F(s) - sf(0^-) - \dot{f}(0^-)$
n^{th} Derivative	$\frac{d^nf(t)}{dt^n} \overset{\mathcal{L}}{\leftrightarrow} s^nF(s) - \sum_{i=1}^{n} s^{n-i}f^{(i-1)}(0^-)$
Integration	$\int_0^t f(\lambda)d\lambda \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}F(s)$
Multiplication by time	$tf(t) \overset{\mathcal{L}}{\leftrightarrow} -\frac{dF(s)}{ds}$
Time Shift	$f(t-a)\gamma(t-a) \overset{\mathcal{L}}{\leftrightarrow} e^{-as}F(s)$ $(\gamma(t) = \text{unit step function})$
Complex Shift	$f(t)e^{-at} \overset{\mathcal{L}}{\leftrightarrow} F(s+a)$
Time Scaling	$f\left(\frac{t}{a}\right) \overset{\mathcal{L}}{\leftrightarrow} aF(as)$
Convolution ("*" denotes convolution of functions)	$f_1(t) * f_2(t) \overset{\mathcal{L}}{\leftrightarrow} F_1(s)F_2(s)$
Initial Value Theorem	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
Final Value Theorem	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Properties of z-Transform

Property Name	Illustration
Linearity	$af_1[k]+bf_2[k] \stackrel{\mathcal{Z}}{\longleftrightarrow} aF_1(z)+bF_2(z)$
Shift Left by 1	$f[k+1] \stackrel{\mathcal{Z}}{\longleftrightarrow} zF(z)-zf[0]$
Shift Left by 2	$f[k+2] \stackrel{\mathcal{Z}}{\longleftrightarrow} z^2F(z)-z^2f[0]-zf[1]$
Shift Left by n	$f[k+n] \stackrel{\mathcal{Z}}{\longleftrightarrow} z^nF(z)-z^n\sum_{k=0}^{n-1} f[k]z^{-k}$ $= z^n\left(F(z)-\sum_{k=0}^{n-1} f[k]z^{-k}\right)$
Shift Right by n	$f[k-n] \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-n}F(z)$
Multiplication by time	$kf[k] \stackrel{\mathcal{Z}}{\longleftrightarrow} -z\dfrac{dF(z)}{dz}$
Convolution	$f_1[k]*f_2[k] \stackrel{\mathcal{Z}}{\longleftrightarrow} F_1(z)F_2(z)$
Initial Value Theorem	$f[0]=\lim_{z\to\infty} F(z)$
Final Value Theorem	$\lim_{k\to\infty} f[k]=\lim_{z\to 1}(z-1)F(z)$